JOURNAL OF COMBINATORIAL THEORY (A) 21, 240–244 (1976)

Note

On Permanents and the Zeros of Rook Polynomials

ALBERT NIJENHUIS

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19174

Communicated by the Managing Editors

Received December 17, 1975

The concept of rook polynomial of a "chessboard" may be generalized to the rook polynomial of an arbitrary rectangular matrix. A conjecture that the rook polynomials of "chessboards" have only real zeros is thus carried over to the rook polynomials of nonnegative matrices. This paper proves these conjectures, and establishes interlacing properties for the zeros of the rook polynomials of a positive matrix and the matrix obtained by striking any one row or any one column.

1. INTRODUCTION

By a "board" B we mean a finite subset of $N \times N$, where $N = \{1, 2, \ldots\}$. For a given board B let r_k denote the number of ways of arranging k "nonattacking rooks" on B: k-subsets of B, no two elements in the same row or column. The polynomial $R(x, B) = r_0 + r_1 x + r_2 x^2 + \cdots$ is called the rook polynomial of B, and it has been extensively studied (e.g., [1, 3]). In all known cases it has been found that $R(x, B)$ has real zeros only, and Goldman, Joichi, and White [1] have conjectured that this is always the case. We prove this conjecture here, using an unpublished idea of H. S. Wilf, generalizing the concept of rook polynomial to the case of an arbitrary matrix instead of a board and then proving that if the matrix has nonnegative entries its rook polynomial has real zeros only.

DEFINITION. Let A be an $m \times n$ matrix. For each $k = 0, 1, 2, \ldots$ define $r_k(A)$ as the sum of all $k \times k$ permanental minors of A; $r_0(A) = 1$.

DEFINITION. By the (alternating) *rook polynomial* of an $m \times n$ matrix A we mean

$$r(x, A) = \sum_{k=0}^{\infty} r_k(A)(-x)^k; \tag{1.1}$$

if A is degenerate ($m = 0$ or $n = 0$) we set $r(x, A) = 1$.

The rook polynomial is insensitive to a permutation of rows or columns, to transposition, and to the adjoining or deleting of a row or column of zeros. The real zeros of the rook polynomial of a nonnegative matrix are necessarily positive.

It is easy to check that, if a board B is identified with a (0, 1) matrix A, the "squares" of B corresponding to the 1-positions in A, then $R(x, B) = r(-x, A)$.

THEOREM 1. *The zeros of the rook polynomial of a matrix of nonnegative entries are real.*

This result implies numerous inequalities for the $r_k(A)$ of nonnegative A, cf. [2].

DEFINITION. By the *rook values* of a matrix A we mean the zeros of its rook polynomial $r(x, A)$.

THEOREM 2. *The rook values of a positive matrix all have multiplicity one.*

THEOREM 3. *Let A be a positive matrix, let A' be obtained from A by the deletion of one row or one column. Then the rook values of A and A' are all real, say $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_{n'}$, respectively, and $n' = n - 1$ or $n' = n$ while*

$$0 < x_1 < y_1 < \cdots < y_{n-1} < x_n \qquad (< y_n \text{ if } n' = n). \tag{1.2}$$

Theorem 1 follows from Theorem 3 by a familiar continuity argument; Theorem 2 is a corollary of Theorem 3, and Theorem 3 is proved through an inductive argument.

The author is pleased to acknowledge several stimulating conversations with H. S. Wilf. It was he who generalized the concept of rook polynomial to a general matrix, and he conjectured Theorem 1 in its present general form, besides establishing (2.1, 3).

2. Separation Properties

Let $f(x)$ and $g(x)$ be real polynomials of respective degrees n and n'. The let us say that property $S(f, g)$ holds if

(a) $n' = n - 1$ or $n' = n$;

(b) $f(x)$ and $g(x)$ have only real zeros, say $x_1, \dots, x_n$ and $y_1, \dots, y_{n'}$ respectively, in increasing order, and the inequalities (1.2) hold;

(c) $f(0) > 0$ and $g(0) > 0$.

Lemma 1. *Let $f(x)$ and $g_1(x), \dots, g_k(x)$ be real polynomials; let $S(f, g_i)$ hold for $i = 1, \dots, k$. If $c_1, \dots, c_k$ are positive, and $F(x) = f(x) - x \sum c_i g_i(x)$, then $S(F, f)$ holds.*

Proof. Hypothesis (c) and the inequalities (1.2) for $S(f, g_i)$ imply that $\operatorname{sign}(g_i(x_j)) = (-1)^{j+1}$. As $f(x_j) = 0$, $x_j > 0$ and $c_i > 0$, this implies $\operatorname{sign}(F(x_j)) = (-1)^j$. Hence, the numbers $x_1, \dots, x_n$ are interlaced with $n - 1$ zeros of $F(x)$. As, furthermore, $F(0) > 0$ and $F(x_1) < 0$, $F(x)$ has at least one zero in $(0, x_1)$. This accounts for n zeros of $F(x)$. If $\deg g_i = n - 1$ for all $i = 1, \dots, k$, these are all zeros of $F(x)$, and the proof is complete. If $\deg g_i = n$ for at least one value of i, then $\deg F = n + 1$, as the positivity of the roots of the g_i and f together with property (c) implies that the coefficient of x^l in g_i and f is zero or has the sign of $(-1)^l$. Since $\operatorname{sign} F(x_n) = (-1)^n$ but $\operatorname{sign} F(x) = (-1)^{n+1}$ as $x \to \infty$, $F(x)$ has another zero in (x_n, ∞).

Let A be an $m \times n$ matrix and c the element in the (i, j) position. Denote by $A_{(i,j)}$ the matrix obtained from A by inserting 0 into the (i, j) position, and by $A_{i|j}$ the matrix obtained from A by striking row i and column j. The $k \times k$ permanental minors which constitute $r_k(A)$ can then be split into two families: those which do not involve the (i, j) position, and those that do. The former are entirely accounted for by terms of $r_k(A_{(i,j)})$. The latter equal c times the permanent of the $(k - 1) \times (k - 1)$ matrix obtained by striking row i and column j. Hence,

$$r_k(A) = r_k(A_{(i,j)}) + c\ r_{k-1}(A_{i|j}), \tag{2.1}$$

and

$$r(x, A) = r(x, A_{(i,j)}) + (-x)\, c\ r(x, A_{i|j}). \tag{2.2}$$

These all generalize known formulas (see, e.g., [3, p. 168f]).

Let A be as above, let $A_{i|}$ be obtained from A by striking row i, and let

$A_{|j}$ be obtained from A by striking column j. Then repeated application of (2.2) to the elements of row i gives

$$r(x, A) = r(x, A_{i|}) - x \sum_{j=1}^{n} a_{ij} r(x, A_{i|j}). \tag{2.3}$$

Similarly, repeated application of (2.2) to the elements of column j gives

$$r(x, A) = r(x, A_{|j}) - x \sum_{i=1}^{m} a_{ij} r(x, A_{i|j}). \tag{2.4}$$

If A is a positive matrix, then so are $A_{i|}$, $A_{|j}$, and $A_{i|j}$, so (2.3) and (2.4) are relations between positive matrices.

PROPOSITION $R(m, n)$. *For any positive $m \times n$ matrix A and for every i, $1 \leqslant i \leqslant m$, property $S(r(x, A), r(x, A_{i|}))$ holds.*

PROPOSITION $C(m, n)$. *For any positive $m \times n$ matrix A and for every j, $1 \leqslant j \leqslant n$, property $S(r(x, A), r(x, A_{|j}))$ holds.*

LEMMA 2. *Let $m, n > 0$, then $C(m, n) \Rightarrow R(m + 1, n)$, $R(m, n) \Rightarrow C(m, n + 1)$.*

Proof. The second statement is obtained from the first by transposition, so it suffices to show the first. Let A^* be a positive $(m + 1) \times n$ matrix. Let the elements in row i be $c_1, \ldots, c_n$, and let A be obtained from A^* by striking row i. Let $A_{|j}$ be obtained, as above, by striking column j from A. Then, by (2.3) we have

$$r(x, A^*) = r(x, A) - x \sum_{j=1}^{n} c_j r(x, A_{|j}).$$

Setting $f(x) = r(x, A)$, $g_j(x) = r(x, A_{|j})$, properties $S(f, g_j)$ hold by hypothesis. By Lemma 1, this implies that $S(F, f)$ holds, where

$$F(x) = f(x) - x \sum c_j g_j(x) = r(x, A^*).$$

As row i of A^* was arbitrary, this proves $R(m + 1, n)$.

LEMMA 3. *Let $m, n > 0$, then $R(1, n)$, $R(m, 1)$, $C(1, n)$ and $C(m, 1)$ hold.*

Proof. The third and fourth statements are obtained from the first two by transposition, and it suffices to prove $R(1, n)$ for $n \geqslant 1$ and $R(m, 1)$ for $m > 1$. Property $R(1, n)$ says that the zero of the rook polynomial $1 - x(a_1 + \cdots + a_n)$ of a row vector is positive, which is trivial. Property $R(m, 1)$ says that the zero of $1 - x(a_1 + \cdots + a_m)$ is positive and less than the zero of $1 - x(a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_m)$, again trivial.

LEMMA 4. *Let $m_0, n_0 > 0$ be such that $R(m, n)$ and $C(m, n)$ hold for all m, n such that $1 \leqslant m \leqslant m_0$ and $1 \leqslant n \leqslant n_0$. Then $C(m_0 + 1, n_0)$ and $R(m_0, n_0 + 1)$ hold.*

Proof. The second statement follows from the first by transposition, so we prove only the first. If $n_0 = 1$, Lemma 3 provides a proof. If $n_0 > 1$, then $C(m_0, n_0 - 1)$ holds by hypothesis, and by Lemma 2, $R(m_0 + 1, n_0 - 1)$ follows. Again by Lemma 2, this implies $C(m_0 + 1, n_0)$.

Theorem 3, which asserts that $R(m, n)$ and $C(m, n)$ hold, now follows from Lemmas 2, 3, and 4, as these provide both the initial phase and the induction step. Theorem 2 is a corollary of Theorem 3. To prove Theorem 1, let A be a nonnegative matrix, ϵ a positive number, and A_ϵ the matrix obtained from A by adding ϵ to each element. Then A_ϵ has real rook values. Consider the rook values as points on the Gaussian sphere, then they depend continuously on ϵ in a well understood sense. As the rook values of A_ϵ lie on the (closed) extended real axis $\mathbb{R} \cup \{\infty\}$, so do those of A.

Note added in proof. The author is indebted to E. Bender for pointing out that the positivity of the rook values of a nonnegative matrix follows from a result by Heilmann and Lieb [4]. These authors associate a polynomial to every graph with weighted edges, and prove the reality of its zeros if the weights are nonnegative. The result on rook values is obtained by observing that the rook polynomial of an $m \times n$ matric is the same as their polynomial for a weighted $K_{m,n}$.

REFERENCES

1. J. R. GOLDMAN, T. J. JOICHI, AND D. E. WHITE, Rook Theory V: Rook polynomials, Möbius inversion and the umbral calculus, *J. Combinatorial Theory, Ser. A.* **21** (1976), 230–239.
2. G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, "Inequalities," The University Press, Cambridge, 1934.
3. J. RIORDAN, "An Introduction to Combinatorial Analysis," Wiley, New York, 1958.
4. O. J. HEILMANN AND E. H. LIEB, Theory of monomer–dimer systems, *Commun. Math. Phys.* **25** (1972), 190–232.